lites. With orbit control 13 encounters fall within 100,000 km of the satellites.

There are two difficulties associated with the suggested periapse impulse control policy which bear mentioning. Firstly, the amount of control impulse used during any one periapse passage is determined from the anticipated perturbations during the next orbit revolution. Obviously, the perturbations estimate is directly affected by the accuracy of the satellite ephemerides used to determine the perturbations. The sensitivity of this estimate to such errors has not been assessed.

Secondly, the control maneuver must be performed at or near periapse (for maximum effect) where it is most likely to interfere with science experiment operations of the spacecraft. Most of the science instruments of a Jupiter orbiter which are concerned with the planet itself, e.g., imagers, spectrophotometers, and radiometers, provide their best resolution data near periapse. Added to this is the fact that some of the Io encounters occur near periapse for the more practical Mode 3 and Mode 7 orbits. Solutions to this conflict of spacecraft duties, such as turning off planetology experiments for the entire period of encounters or not performing a control impulse on every orbit, have not been analyzed. Both of these problems should be considered in future more detailed studies of the multiple satellite encounter technique.

## Conclusions

It is concluded that practical Jupiter orbits are available which provide multiple observations of all four Galilean satellites. Orbit radiation lifetime and capture impulse requirements suggest the use of 14.222-day orbits together with either the Mode 3 or Mode 7 encounter sequences. This combination typically provides 35 close encounters (maximum satellite disc size larger than the moon as viewed from earth) in a period of 170 days. These encounters occur with a frequency of about one every 5 days. The approach direction is always from the sun, which eases the task of locating the satellite well before closest approach. More than 6 hr during flyby are available for surface imagery at variable conditions of solar illumination and resolution. With proper orbit control radio occultations should be possible during at least one encounter with each satellite within a range of one Jupiter radius (~70,000 km).

Initial orbit conditions must be achieved accurately to follow the predicted encounter profiles. Errors in orbit period are particularly unacceptable. Once the encounters begin it will also be necessary to perform small but essential orbit control impulses to counter the disruptive effects of satellite perturbations on the reference orbit.

## References

[1] Niehoff, J. C. et al., "First Generation Jupiter Orbiter Missions," Astro Sciences Rept. M-20, IIT Research Inst., Chicago, Ill., to be published.

[2] Melbourne, W. G. et al., "Constants and Related Information for Astrodynamics Calculations, 1968," Tech. Rept. 32-1306, 1968, Jet Propulsion Lab., Pasadena, Calif.

[3] Sampson, R. A., "Theory of the Four Great Satellites of Jupiter," *Memoirs of the Royal Astronomical Society*, Vol. LXIII, 1920.

[4] Niehoff, J. C., "Touring the Galilean Satellites," AIAA Paper 70-1070, Santa Barbara, Calif., Aug. 1970.

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# Switching Conditions and a Synthesis Technique for the Singular Saturn Guidance Problem

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**A singular optimal guidance problem which was motivated by difficulties encountered in the Saturn V AS-502 flight has been studied. It is shown that if the guidance equations are based upon a singular version of the flat-Earth problem, then the control must be discontinuous at a junction of singular and nonsingular subarcs for almost all cases. A good suboptimal guidance scheme based upon a nonsingular approximation of the singular problem is presented. The resultant suboptimal control is continuous, which is more desirable then a discontinuous control, and causes only a noise-level difference in payload.**

## Introduction

IN the second flight of the Saturn V vehicle (AS-502), two engines shut down early in the S-II stage. The measurements received by the onboard guidance scheme, the Iterative Guidance Mode (IGM),[1] indicated that only one engine was out. This resulted in a steep planar steering program in the S-IVB stage which caused the time rate of change of the steering angle to reach its limiting value for a large portion of the S-IVB flight. Since the IGM is based on unconstrained variational theory the resultant trajectory did not reach the desired terminal orbit.

In the aforementioned flight a large disturbance caused the guidance law to determine a steering angle rate of change which was too large. Thus, it would be desirable to design the guidance logic in such a way that the time rate of change of the steering angle is a bounded control variable, say $u$ with $|u| \leq K$, and such that the steering angle is a state variable since it cannot change rapidly (because of physical and reliability constraints). However, the resultant optimal control problem is a singular problem, and the variational and computational theory for such problems is far from satisfactory.

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Recently, new necessary conditions and sufficient conditions for totally singular problems [i.e., problems such that Eq. (9) below holds almost everywhere] have been developed.[2,3,4] Thus, the main problems to be resolved are: 1) the determination of necessary and sufficient conditions for optimal trajectories which possess both singular and nonsingular subarcs (which is the case in the Saturn guidance problem); and 2) the development of a computational scheme for the generation of optimal trajectories which possess both singular and nonsingular subarcs. Partial results in this direction have also been obtained recently.[5,6]

In the following analysis, necessary conditions for composite optimal trajectories (i.e., trajectories which contain both singular and nonsingular subarcs) are used to characterize the local switching behavior of the singular Saturn guidance problem. Since the resultant behavior is not physically desirable, the problem is transformed into a good suboptimal nonsingular representation of the problem which could be incorporated easily into a recently proposed guidance scheme for Saturn class vehicles.[7,8]

## Singular Optimal Control Theory

In this section, properties from singular optimal control theory which are applied later will be summarized.

Consider the problem of minimizing

$$J = G(t_f,x_f) + \int_{t_0}^{t_f} L(t,x)dt \tag{1}$$

subject to the following conditions

$$\dot{x} = f(t,x,u) \equiv f^0(t,x) + f_u(t,x)u \tag{2}$$

$$x(t_0) = x_0, \psi(t_f,x_f) = 0, |u| \leq K \tag{3}$$

The state $x$ is $n$-dimensional, $u$ is a scalar control variable, and $\psi$ is a $p$-dimensional vector function which defines the terminal surface, $p \leqq n + 1$.

Along an optimal trajectory the following necessary conditions hold[9]:

$$\dot{\lambda} = H_x(t,x,\lambda,u) \tag{4}$$

$$\lambda^T(t_f) = G_{x_f}{}^T(t_f,x_f) + \nu^T\psi_{x_f}(t_f,x_f) \tag{5}$$

$$H(t_f,x(t_f),\lambda(t_f),u(t_f)) + G_{t_f}(t_f,x_f) + \nu^T\psi_{t_f}(t_f,x_f) = 0 \tag{6}$$

$$H(t,x,\lambda,u) = \min_{|v| \leqq K} H(t,x,\lambda,v) \tag{7}$$

where $\lambda(t)$ and $\nu$ are Lagrange multipliers and

$$H(t,x,\lambda,u) \equiv L(t,x) + \lambda^T f(t,x,u) \tag{8}$$

In general, the optimal trajectory for this problem consists of some combination of singular arcs and nonsingular (bang-bang) arcs. A singular arc is one along which

$$H_u(t,x,\lambda) \equiv 0 \tag{9}$$

on a nonzero time interval. A nonsingular arc is one along which $H_u(t,x,\lambda) \neq 0$ except possibly at a countable number of points $\{t_1,t_2,\ldots\} \subset [t_0,t_f]$. Since the Pontryagin minimum principle is satisfied trivially along a singular subarc, the following condition is useful for classifying singular arcs, and for distinguishing between maxima and minima.

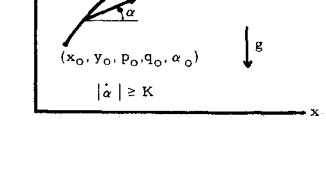


**Fig. 1 The singular flat-Earth problem.**

**Kelley Condition**[10]

Let $u$ be an optimal singular control on $[t_1,t_2] \subset [t_0,t_f]$, and let $H_u{}^{(2q)}$ be the lowest order total time derivative of $H_u$ in which $u$ appears explicitly with a coefficient which is nonzero almost everywhere on $[t_1,t_2]$. Then, the integer $q$ is called the order of the singular arc, and

$$(-1)^q \, \partial H_u{}^{(2q)}/\partial u \geq 0 \tag{10}$$

In Ref. 11, Taylor series expansions in the neighborhood of a singular-nonsingular junction along with the maximum principle are used to obtain necessary conditions at the junction. Recently these results have been generalized[12] as follows.

**Theorem 1**

Let $x(t)$ be an optimal trajectory which contains both singular and nonsingular subarcs, and let the singular subarcs be of the $q$th order, i.e.,

$$H_u{}^{(2q)} \equiv A(t,x,\lambda) + B(t,x,\lambda)u \tag{11}$$

Suppose the optimal control is piecewise analytic in a neighborhood of a junction (this is not always the case as is shown in Ref. 11), and $B(t,x,\lambda) \neq 0$ at the junction. If $u^{(r)} \equiv d^r u/dt^r$ ($r \geqq 0$, where $u^{(0)} \equiv u$) is the lowest-order derivative of $u$ which is discontinuous at the junction, then $q + r$ must be an odd integer.

The main consequence of this theorem is that if a Taylor series expansion is valid in the neighborhood of a junction and the control is discontinuous at the junction, then the singular subarc must be of odd order. The singular Saturn guidance problem contains odd order ($q = 1$) singular subarcs. Note that the theorem does not imply that the optimal control must jump if $q$ is odd. Also, there exist well-known cases of $q$-even problems with discontinuous controls but the controls are not piecewise analytic (e.g., an infinite number of switches between $u = \pm K$ on the nonsingular side of the junction in a finite time interval).

With the analyticity assumption removed, the following result can be obtained:

**Theorem 2**

Let $u$ be an optimal control which contains both nonsingular and piecewise continuous, $q$th order singular subarcs. 1) If $H_u{}^{(2q)} \neq 0$ on the nonsingular side of a junction, then the control is discontinuous. 2) If $A = 0$, $B \neq 0$, and $K \neq 0$ at a junction, then the control is discontinuous. 3) If $u$ is piecewise continuous on the nonsingular subarc, $H_u{}^{(2q)} = 0$ on the nonsingular side of a junction, and $B \neq 0$ at the junction, then the control is continuous.

Theorem 2 has the desirable quality of determining if the control "jumps" or is continuous at a junction without an analyticity assumption. However, the conditions are more difficult to verify than those of Theorem 1 if analyticity is a valid assumption. In some cases the theorems can be used together to indicate what one cannot assume, e.g., $H_u{}^{(2q)} \neq 0$ on the nonsingular side of the junction and $q$ even imply that $u$ is discontinuous by Theorem 2, but by Theorem 1, $u$ must be continuous if it is piecewise analytic. Thus, one may conclude that if a junction occurs it is a nonanalytic junction.

## Saturn Guidance: A Singular Flat-Earth Problem

In this section, an analysis of the switching procedure for a flat-Earth representation of circular orbit insertion will be presented. This is directly applicable to the guidance of the Saturn V vehicle since the IGM is based upon the flat-Earth approximation. It will be shown that except for an exceptional case, the optimal control is discontinuous at a junction of singular and nonsingular subarcs. Thus, the optimal control does not "ride" onto or off of the control boundary.

The planar equations of motion and boundary conditions for the singular flat-Earth problem are (see Fig. 1):

$$\begin{aligned} \dot{x} &= p & x(t_0) &= x_0 \\ \dot{y} &= q & y(t_0) &= y_0, y(t_f) = r_c \\ \dot{p} &= (F/m)\cos\alpha & p(t_0) &= p_0, p(t_f) = v_c \\ \dot{q} &= (F/m)\sin\alpha - g & q(t_0) &= q_0, q(t_f) = 0 \\ \dot{\alpha} &= u & \alpha(t_0) &= \alpha_0, |u| \leqq K \\ m(t) &= m_0 + \dot{m}_0(t - t_0) \end{aligned} \tag{12}$$

where $r_c \equiv$ radius of the circular orbit, $v_c \equiv$ circular velocity at $r_c$. It is desired to transfer the vehicle from the given initial conditions into the given circular orbit in minimum time.

The Hamiltonian is

$$H = \lambda_1 p + \lambda_2 q + \lambda_3(F/m)\cos\alpha + \lambda_4[(F/m)\sin\alpha - g] + \lambda_5 u \tag{13}$$

which implies, by Eqs. (4–7)

$$\begin{aligned} \lambda_1(t) &\equiv 0 & \lambda_3(t) &\equiv c_3 \\ \lambda_2(t) &\equiv c_2 & \lambda_4(t) &= c_4 - c_2 t \\ \dot{\lambda}_5 &= (F/m)(\lambda_3\sin\alpha - \lambda_4\cos\alpha), & \lambda_5(t_f) &= 0 \end{aligned} \tag{14}$$

If an optimal singular subarc exists, then by the Kelley condition,

$$\dot{H}_u = (F/m)(\lambda_3\sin\alpha - \lambda_4\cos\alpha) \tag{15}$$

$$\ddot{H}_u = (F/m)(\lambda_2\cos\alpha - \lambda_1\sin\alpha) + (F/m)(\lambda_3\cos\alpha + \lambda_4\sin\alpha)u \equiv A(t,x,\lambda) + B(t,x,\lambda)u \tag{16}$$

implies

$$B(t,x,\lambda) \leqq 0 \quad \text{(on singular arc)} \tag{17}$$

By Eq. (15), i.e., $\dot{H}_u \equiv 0$ on a singular subarc,

$$\tan\alpha = \lambda_4/\lambda_3 \rightarrow \begin{cases} \cos\alpha = \pm\lambda_3(\lambda_3^2 + \lambda_4^2)^{-1/2} \\ \sin\alpha = \pm\lambda_4(\lambda_3^2 + \lambda_4^2)^{-1/2} \end{cases} \tag{18}$$

and by Eq. (17)

$$B(t,x,\lambda) = \pm(F/m)[\lambda_3^2 + \lambda_4^2]^{1/2} \leqq 0 \tag{19}$$

which implies that the minus sign should be chosen in Eq. (18) for the minimum time problem. Upon substitution of Eqs. (14) and (18) into Eq. (16)

$$\ddot{H}_u = -(F/m)c_2c_3(c_3^2 + \lambda_4^2)^{-1/2} + (F/m)(c_3^2 + \lambda_4^2)^{1/2}u \quad \text{(on singular arc only)} \tag{20}$$

We shall now consider under what conditions a saturation junction is possible, i.e., the control rides on or off of the boundary (or, is continuous at the junction).

If the control is continuous and well-behaved at the junction, then Theorem 1 is applicable, i.e., the control is piecewise analytic in a neighborhood of the junction. Since $q = 1$ for this problem, if the control is assumed to be continuous at the junction, then by Theorem 1, $r \geqq 2$ (i.e., if $r \neq 0$, then $r \neq 1$ since $q + r$ must be odd). Thus, if $u$ is continuous, then $\dot{u} = \ddot{\alpha}$ is continuous, also. By Eq. (18), the expression for $\ddot{\alpha}$ on the singular arc may be determined

$$\dot{u} = \ddot{\alpha} = 2c_2^2c_3\lambda_4/(c_3^2 + \lambda_4^2)^2 \quad \text{(on singular arc)} \tag{21}$$

Since $\dot{\alpha} = \pm K$ on the nonsingular arc, it follows that

$$\dot{u} = \ddot{\alpha} = 0 \quad \text{(on nonsingular arc)} \tag{22}$$

Therefore, the continuity of $\dot{u}$ at the junction requires $\dot{u} = 0$ at the junction, which implies

$$c_2 = 0 \text{ or } c_3 = 0 \text{ or } \lambda_4 = 0 \tag{23}$$

If $c_2 = 0$ or $c_3 = 0$, then the first term in Eq. (20) is zero, which implies $u$ is discontinuous at the junction by Theorem 2 (2). Thus, $c_2 \neq 0$, $c_3 \neq 0$ if the junction is continuous, and Eq. (23) then implies,

$$\lambda_4 = 0 \quad \text{(at a continuous junction)} \tag{24}$$

or by Eq. (18),

$$\alpha = 0°, 180° \quad \text{(at a continuous junction)} \tag{25}$$

i.e., if $\alpha \neq 0°, 180°$ at a junction, then the junction must be discontinuous. Since the steering angles $\alpha = 0°, 180°$ do not appear to possess special properties, further analysis would probably eliminate the possibility of a continuous junction at these points, also.

If indeed smooth junctions are possible when $\alpha = 0°, 180°$, then one can easily show that only one smooth junction is possible on the trajectory since $\lambda_4$ is a linear function of time and, thus, can go through zero only once. Also, if one considers an inverse-square gravity field and it is assumed that a continuous junction is possible, then necessary conditions for such a junction can be derived in the same way as Eq. (24) was derived for the flat-Earth problem which was previously discussed.

## Synthesis of Guidance Laws for Singular Problems

In the previous section it was shown that a junction of singular and nonsingular subarcs in the singular flat-Earth problem requires a jump in the control (except possibly for the zero-probability cases when $\alpha = 0°, 180°$). Since the time to jump is mainly a function of nonlocal information, a formidable synthesis problem arises. In this section a suboptimal synthesis procedure to be used in conjunction with the guidance scheme of Refs. 7 and 8 is suggested.

In Refs. 7 and 8, a guidance scheme based upon the onboard solution of a nonsingular two-point boundary-value problem is proposed. Such a scheme is possible for Saturn class vehicles since it has a relatively large onboard computer. In this section a nonsingular approximation of the singular Saturn guidance problem will be developed, and the flat-Earth approximation will be relaxed. The resultant optimal time rate of change of the steering angle (i.e., the optimal control) has the desirable property of continuity.

The planar equations of motion and boundary conditions for the singular inverse-square problem in polar coordinates are (see Fig. 2):

$$\begin{aligned} \dot{r} &= v_r & r(t_0) &= r_0, r(t_f) = r_c \\ \dot{\theta} &= v_\theta/r & \theta(t_0) &= \theta_0 \\ \dot{v}_r &= v_\theta^2/r - \mu/r^2 + (F/m)\sin\gamma & v_r(t_0) &= v_{r0}, v_r(t_f) = 0 \\ \dot{v}_\theta &= v_r v_\theta'/r + (F/m)\cos\gamma & v_\theta(t_0) &= v_{\theta 0}, v_\theta(t_f) = v_c \\ \dot{\gamma} &= u & \gamma(t_0) &= \gamma_0, |u| \leqq K \\ m(t) &= m_0 + \dot{m}_0(t - t_0) \end{aligned} \tag{26}$$

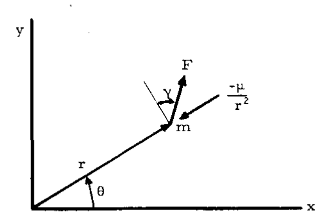


**Fig. 2 Coordinate system and steering angle for the inverse-square gravity problem.**

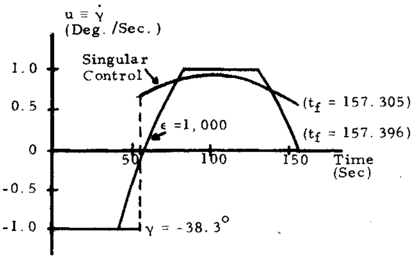


**Fig. 3 Optimal singular control and $\epsilon$ = 1,000 suboptimal control.**

where

$$J_1 = t_f \tag{27}$$

is to be minimized.

In Ref. 5, the following method for computing singular control problems is suggested: adjoin

$$\epsilon \int_{t_0}^{t_f} u^2 dt$$

to the performance index, i.e., Eq. (27), and solve the resultant nonsingular boundary value problem for successively smaller values of $\epsilon$. A $\epsilon \to 0$, it is proved that the solutions approach the optimal singular solution. Another computational scheme is also suggested in Ref. 5 since numerical stability problems may result for small values of $\epsilon$. However, the main effect of the second scheme is to sharpen the control history while only a slight improvement in the performance index is noted. Data from the Saturn AS-502 flight will be used to show that merely adding the

$$\epsilon \int_{t_0}^{t_f} u^2 dt$$

term to the performance index results in a good suboptimal control for a relatively large value of $\epsilon$.

Define

$$J_2 = t_f + \epsilon \int_{t_0}^{t_f} u^2 dt \tag{28}$$

where $\epsilon$ is a given constant. The Hamiltonian is

$$H = \lambda_1 v_r + \lambda_2 v_\theta / r + \lambda_3 [v_\theta^2/r - \mu/r^2 + (F/m)\sin\gamma] + \lambda_4[-v_r v_\theta/r + (F/m)\cos\gamma] + \lambda_5 u^2 + \epsilon u^2 \tag{29}$$

which defines a nonsingular optimization problem. The minimum principle states that the Hamiltonian must be minimized with respect to the control. This implies the following ordinary minimization problem: minimize

$$\tilde{h} \equiv \lambda_5 u + \epsilon u^2 \tag{30}$$

subject to the inequality constraint

$$|u| \leqq K \tag{31}$$

Eq. (31) can be transformed into an equality constraint by introducing a slack variable $z$, i.e.,

$$z^2 = K - u^2 \geqq 0 \tag{32}$$

is an equality constraint which enforces the desired inequality constraint. By defining the augmented function

$$h(u,z) = \lambda_5 u + \epsilon u^2 + \Lambda(z^2 + u^2 - K) \tag{33}$$

and forming

$$\partial h/\partial u = 0, \partial h/\partial z = 0 \tag{34}$$

and then checking the second-order sufficient condition for ordinary minimization problems, the following optimal control is determined:‡

$$u = \begin{cases} -K & \text{if } \lambda_5 \geqq 2\epsilon K \\ -\lambda_5/2\epsilon & \text{if } -2\epsilon K \leqq \lambda_5 \leqq 2\epsilon K \\ +K & \text{if } \lambda_5 \leqq -2\epsilon K \end{cases} \tag{35}$$

Note that the control is continuous at the junction points $\lambda_5 = \pm 2\epsilon K$ since $\lambda_5$ must be continuous by the Weierstrass-Erdmann corner conditions.

The usual Euler-Lagrange equations hold for the multipliers. The only other new condition of interest is the transversality condition for $\lambda_5(t_f)$. Since $\gamma(t_f)$ is unspecified, then

$$\lambda_5(t_f) = 0 \tag{36}$$

which implies that

$$-2\epsilon K < \lambda_5(t_f) < 2\epsilon K, \text{ or}$$
$$u(t_f) = -\lambda_5(t_f)/2\epsilon = 0 \tag{37}$$

This states that the control must have an interior segment in a neighborhood of $t_f$. However, in some numerical studies for $\epsilon$ small this terminal interior arc was very short, e.g., 0.1 sec of a 160 sec trajectory.

Since the guidance scheme of Refs. 7 and 8 involves an iteration scheme which uses initial Lagrange multiplier estimates, a similar scheme was used to converge the optimal trajectories of this study. (Since a sufficient condition for composite singular problems does not exist, we can only use physical reasoning to argue that the resultant extremals are indeed optimal.)

The data listed in Appendix A represent the state (position, velocity, and weight) of the SIVB stage of the Saturn SA-502 flight at approximately 160 sec before cutoff. The steering angle at this instant, say $t_0$, was $\gamma(t_0) = +17.1°$. Using unconstrained optimization theory (i.e., assuming $\gamma$ can change instantaneously), the onboard guidance system computed the "best" steering angle to guide the vehicle into the desired circular orbit. The value obtained was approximately $\gamma = -29.5°$, which would require an instantaneous change of 46.6° in the steering angle. Since the steering rate on the Saturn is constrained to approximately one degree per second, the required steering angle could not be attained.

By reformulating the problem as a bounded control problem with the control $u = \dot{\gamma}$, $|u| \leq 1°$/sec., the optimal trajectory was computed. In Fig. 3, the optimal time rate of change of the steering angle is presented. The optimal control is nonsingular on the interval [0, 55.4) and singular on the interval (55.4, 157.305]. Note that the control is discontinuous at the junction, which is expected since the flat-Earth problem is an excellent approximation of the problem of this section. The difference in the initial value of $\gamma$ causes the constrained trajectory to be approximately 6.5 sec longer than the trajectory computed with $\gamma(t_0) = -29.5°$, which results in a 3500 pound fuel loss.

The optimal control for a nonsingular approximation of the given singular problem is presented in Fig. 3, also. The value $\epsilon = 1{,}000$ was found to give good results with respect to optimality and ease of convergence. Note that the suboptimal control is continuous, and in some sense approximates the optimal singular control. The final time of the suboptimal trajectory is 157.396, which represents a fuel penalty of only 48.8 lbs.

The iteration scheme employed to converge the trajectories in Fig. 3 was a shooting technique (i.e., an initial Lagrange multiplier iteration procedure) based upon the secant method. In the computation of the $\epsilon$ − approximate trajectories, it was found that the initial Lagrange multipliers were very close to linear functions of $\epsilon$ on the $\epsilon$-interval [$10^3$, $10^5$] and, thus,

‡ Equations (35) can be determined alternatively by adjoining $\mu(u^2 - K)$ to the Hamiltonian (29) and imposing the Clebsch necessary condition $\mu \geq 0$ (see pp. 108–109, Ref. 9).

the generation of suboptimal trajectories defined by the sequence $\{\epsilon = 1 \times 10^5, 5 \times 10^4, 2 \times 10^4, 1 \times 10^4, 1 \times 10^3\}$ was rapid. In fact the suboptimal trajectories are qualitatively the same on this interval with: $t_1(\epsilon = 10^3) = 41.695$ and $t_1(\epsilon = 10^5) = 40.081$, $t_2(\epsilon = 10^3) = 83.199$ and $t_2(\epsilon = 10^5) = 82.824$, $t_3(\epsilon = 10^3) = 132.075$ and $t_3(\epsilon = 10^5) = 133.918$, and $t_f(\epsilon = 10^3) = 157.396$ and $t_f(\epsilon = 10^5) = 157.416$, where $t_1$, $t_2$, $t_3$ indicate the switch points (i.e., entry to or exit from the control boundary) of the suboptimal trajectories. The difference in payload between the $\epsilon = 10^3$ and $\epsilon = 10^5$ trajectories is only eleven pounds. Note that the sequence of switch times is indicating convergence to the optimal singular solution since $t_1$ is increasing as $\epsilon$ decreases, and since the subarc on $[t_1, t_f]$ is becoming more interior as $\epsilon$ decreases. (In the limit, the $u = +1°/\text{sec}$ subarc shrinks to zero.)

In Ref. 13, from which this paper is derived, a "puzzling trend" with regard to the convergence to the $\epsilon$-method was discussed. It was noted that in converging the trajectories, the payload increased as $\epsilon$ increased, which was contrary to the basic convergence theorem of the method. However, the convergence theorem was proved for the specified final time problem, and it was argued that variable final time could be the cause of the trend. It is a straightforward exercise to extend the convergence theorem to the variable final time case, and thus, this was not the cause of the problem.

The trend was actually due to the fact that the wrong value of $\epsilon$ was assigned to the computed extremals because of the way the initial Lagrange multipliers were scaled. This behavior is discussed further in Appendix B.

## Conclusions

A singular optimal guidance problem which was motivated by difficulties encountered in the Saturn V AS-502 flight has been studied. It was shown that if the guidance system uses a singular version of the flat-Earth problem, then the control must be discontinuous at a junction of singular and nonsingular subarcs for almost all cases. Since the junctions of singular and nonsingular subarcs are determined by nonlocal information, the situation described is undesirable.

A suboptimal guidance scheme based upon a nonsingular approximation of the singular problem was suggested. Since it allows for rapid computation of a nonsingular two-point boundary-value problem, the scheme could be incorporated into the guidance scheme of Refs. 7 and 8.

## Appendix A

Data from the Saturn AS-502 flight, 586.72 sec into the mission (with approximately 160 sec of flight remaining)

| | |
|---|---|
| $x = 6.2133939 \times 10^6$ m | $y = 2.1301780 \times 10^6$ m |
| $\dot{x} = -2.0242460 \times 10^3$ m/sec | $\dot{y} = 6.4412899 \times 10^3$ m/sec |
| $F = 2.2790300 \times 10^5$ lbs. | $W = 3.5280200 \times 10^5$ lbs. |
| $Isp = 4.2476900 \times 10^2$ sec | $\gamma = +17.1$ deg |
| $r_c = 6.5633660 \times 10^6$ m | $v_c = 7.7930430 \times 10^3$ m/sec |

## Appendix B

In the computation of optimal trajectories with unbounded control variables by shooting methods, the number of unknown initial Lagrange multipliers can usually be reduced by one in free variable final time problems. This property is a consequence of the necessary conditions being homogeneous in the Lagrange multipliers. This is a desirable property since the convergence of shooting methods usually is more rapid for problems with fewer unknown initial multipliers.

For the optimal trajectory problem defined by Eqs. (26) and (28) with a given value of $\epsilon$, four initial multipliers are unknown [$\lambda_2(t_0)$ is known since $\lambda_2(t) \equiv 0$]. Even though this is a variable final time problem, the initial Lagrange multipliers cannot be scaled (which implies one of the initial multipliers may be specified arbitrarily) because the switching function $\lambda_5$ would change and, thus, the trajectory would change. However, to obtain a converged case for a value of $\epsilon$ to be determined, a problem with only three unknown initial multipliers may be defined.

For example, let $\tilde{\epsilon}$ be any positive real number and suppose $\lambda_1(t_0) = 1.0$ is specified for each iterate. The initial multipliers $\lambda_3(t_0)$, $\lambda_4(t_0)$, and $\lambda_5(t_0)$ are unknown. If $v_\theta(t_f) = v_c$ is used as a stopping condition, then satisfaction of the boundary conditions $r(t_f) = r_c$, $v_r(t_f) = 0$, and $\lambda_5(t_f) = 0$ is used to adjust the initial multiplier estimates. Note that the transversality condition associated with the variable final time, $H(t_f) = -1.0$, is not necessarily satisfied on the converged solution.

Suppose $H(t_f) = c \neq 0, -1$ on the converged solution with $\epsilon = \tilde{\epsilon}$ and multipliers $\lambda_i(t)$. To satisfy the remaining transversality condition, the multipliers must be scaled, i.e., define

$$l_i(t) \equiv -\lambda_i(t)/c. \qquad (i = 1, \ldots, 5) \tag{B1}$$

Then, if the terminal state values and $u$ remain unchanged, the terminal value of the Hamiltonian with the new multipliers, say $\tilde{H}$, is

$$\tilde{H}(t_f) = \left[\sum_{i=1}^{5} l_i f_i(t,x,u) + \epsilon u^2\right]_{t=tf} \tag{B2}$$

or

$$\tilde{H}(t_f) = -H(t_f)/c = -1 \tag{B3}$$

where Eq. (37) is used to eliminate the $\epsilon u^2(t_f)$ term.

To guarantee that the terminal values remain unchanged, the switch times on the trajectory defined by the $l_i(t)$ must be the same as on the $\lambda_i(t)$ trajectory. This can be accomplished by using

$$\epsilon = -\tilde{\epsilon}/c \tag{B4}$$

on the $l_i(t)$ trajectory since

$$\lambda_5(t_i) = \pm 2\tilde{\epsilon}K \rightarrow l_5(t_i) = \pm 2K(-\tilde{\epsilon}/c) \tag{B5}$$

where $t_i$ indicates a switch time. Thus, a suboptimal $\epsilon$-trajectory may be determined by guessing only three initial multipliers. In Ref. 13, the three parameter method was used to determine the trajectories but Eq. (B4) was not employed to determine the correct value of $\epsilon$. The optimal controls displayed in Figs. 5 and 6 in Ref. 13 are indeed extremal controls but not for the values of $\epsilon$ noted on the curves.

## References


[1] Chandler, D. C., and Smith, I. E., "Development of the Iterative Guidance Mode with Its Application to Various Vehicles and Missions," *Journal of Spacecraft and Rockets*, Vol. 4, No. 7, 1967, pp. 898–903.

[2] McDanell, J. P., and Powers, W. F., "New Jacobi-Type Necessary and Sufficient Conditions for Singular Optimization Problems," *AIAA Journal*, Vol. 8, No. 8, 1970, pp. 1416–1420.

[3] Speyer, J. L. and Jacobson, D. H., "Necessary and Sufficient Conditions for Optimality for Singular Control Problems: A Transformation Approach," *Journal of Mathematical Analysis and Applications*, Vol. 33, 1971, pp. 163–187.

[4] Goh, B. S., "A Theory of the Second Variation in Optimal Control," unpublished report, Jan. 1970, Division of Applied Mechanics, University of California, Berkeley, Calif.

[5] Jacobson, D. H., Gershwin, S. B., and Lele, M. M., "Computation of Optimal Singular Controls," *IEEE Transactions on Automatic Control*, Vol. AC-15, No. 1, 1970, pp. 67–73.

[6] Pagurek, B. and Woodside, C. M., "The Conjugate Gradient Method for Optimal Control Problems with Bounded Control Variables," *Automatica*, Vol. 4, 1968, pp. 337–349.

[7] Brown, K. R. and Johnson, G. W., "Real-Time Optimal Guidance," *IEEE Transactions on Automatic Control*, Vol. AC-12, No. 5, 1967, pp. 501–506.

[8] Brown, K. R., Harold, E. F., and Johnson, G. W., "Rapid Optimization of Multiple-Burn Rocket Flights," NASA CR-1430, Sept. 1969.

[9] Bryson, A. E. and Ho, Y. C., *Applied Optimal Control*, Blaisdell Publishing Co., Waltham, Mass., 1969.

[10] Kelley, H. J., "A Second Variation Test for Singular Extremals," *AIAA Journal*, Vol. 2, No. 8, 1964, pp. 1380–1382.

[11] Kelley, H. J., Kopp, R. E., and Moyer, H. G., "Singular Extremals," *Topics in Optimization*, edited by G. Leitman, Academic Press, 1967.

[12] McDanell, J. P. and Powers, W. F., "Necessary Conditions for the Joining of Optimal Singular and Nonsingular Subarcs," *SIAM Journal on Control*, Vol. 9, No. 2, May 1971, pp. 161–173.

[13] Powers, W. F. and McDanell, J. P., "Switching Conditions and a Synthesis Technique for the Singular Saturn Guidance Problem," AIAA Paper 70-965, Santa Barbara, Calif., 1970.

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# Dynamically Unbalanced Dual-Spin Space Stations with Rigid or Low-Coupling Interconnections

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**Effects of dynamic unbalances are determined for an axially symmetrical, dual-spin space station consisting of a rotating artificial gravity section and a controlled, despun zero gravity section. Configurations with either a rigid interconnection or a low-coupling flexible interconnection between sections are considered. Amplitudes of motion and control system requirements are compared for these configurations. Both configurations are found to experience coning motions. For large space stations with rigid interconnection, control moment gyros (CMGs) are found impractical for reducing coning motions to within allowable limits for many experiments and special systems such as active mass balancing are necessary. However, for a flexible interconnection, coning motions of the despun section are substantially reduced and CMGs could be utilized to reduce motions to acceptable levels.**

## Introduction

SEVERAL studies[1–3] have been conducted in recent years on the attitude motion of dual-spin spacecraft consisting of two bodies with relative motion restricted to rotations about a common axis fixed in both. However, it has been proposed[4] that a low-coupling interconnection which also permits relative rotations of the bodies about directions normal to that axis may offer advantages over the rigid interconnection. Reference 4 states that a low-coupling interconnection designed to transmit only low-levels of torque normal to the common axis could attenuate the effects of mass unbalances of the spinning section on the motion of the despun section.

A recent study[5] has determined that such isolation does occur and that dissipation in a flexible interconnection is an effective means of attenuating nutation. However, that study considers a passive system except for a constant speed motor driving the spinning section at a fixed relative rate. Following is an analysis of the effects of dynamic mass unbalances on attitude motion and stability for dual-spin satellites that are actively controlled by control moment gyros (CMGs) and have either a rigid interconnection or a low-coupling interconnection. A comparison is then given for amplitudes of motion and CMG requirements for the two interconnections and the results are applied to a large space base.

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## System Description

The space station dynamical model to be considered is shown in Fig. 1 and consists of two sections attached by a massless gimbal arrangement. The satellite is stabilized by CMGs providing three axis control on one of the sections. The other section is assumed driven at a constant rate about a line which is not a principal axis of inertia for that body.

System details and terminology are given as follows: The spinning section is termed body $B$ and the other section is termed body $A$. Both are axially symmetrical with principal moments of inertia for their mass centers denoted by $B_3$ and $A_3$, respectively, in the directions of the symmetry axes, and $B_1$ and $A_1$, respectively, in directions normal to the symmetry axes. For body $B$, the mass is designated $M_B$, the mass center is designated $B^*$, and a right-handed, mutually orthogonal set of unit vectors parallel to principal axes of $B$ at $B^*$ is designated by $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. Similar quantities for body $A$ are mass $M_A$, mass center $A^*$, and unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, parallel to principal axes at $A^*$.

An inertially fixed set of unit vectors $\mathbf{a}_{10}$, $\mathbf{a}_{20}$, $\mathbf{a}_{30}$, is defined by the initial orientation of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ at time $t = 0$. The orientation of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ (and consequently $A$) at some subsequent time $t$ is described with respect to $\mathbf{a}_{10}$, $\mathbf{a}_{20}$, $\mathbf{a}_{30}$, by a 1,2,3 sequence of three axis Euler angles $\phi_1$, $\phi_2$, $\phi_3$. Also, the CMG control torque on $A$ is assumed related to $\phi_1$, $\phi_2$, $\phi_3$, by

$$ {}^C\mathbf{T}^A = -[(K_0\phi_1 + K_1\dot{\phi}_1)\mathbf{a}_1 + (K_0\phi_2 + K_1\dot{\phi}_2)\mathbf{a}_2 + (K_{03}\phi_3 + K_{13}\dot{\phi}_3)\mathbf{a}_3] \quad (1) $$

where $K_0$, $K_{03}$, $K_1$, $K_{13}$ are measures of the level of attitude control.